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Geometry of the Einstein and Yang–Mills Equations

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It is shown that the Einstein and Yang–Mills equations arise from the conditions for the space-time to be a submanifold of a pseudo-Euclidean space with dimension greater than 5. Some possible applications to cosmology, spin-2 fields, and geometrodynamics are discussed.

1. INTRODUCTION

Up to the present there is no experimental evidence or even a consistent theory which supports or excludes the idea that physical space-time is a submanifold of a higher dimensional space, as opposed to the usual, stand-alone, Riemann structure. Nonetheless, the fact that the Einstein and Yang–Mills equations are implicit in the fundamental theorem of submanifolds, stating that any d-dimensional manifold is isometrically and locally embeddable in a D-dimensional space $\mathcal{M}_D$, suggests that the theory of submanifold space-times should be taken seriously. Applications of this result to high-energy physics have been frequently proposed (Friedman, 1961; Joseph, 1962; Fronsdal, 1965; Ne'emann and Rosen, 1965; Bergmann, 1982; Wetterich, 1985; Maia, 1988). The purpose of this note is to examine the physical role of the extrinsic curvature of the space-time in the context of that theorem.

Two basic problems must be solved before the embedding of a space-time could be considered: One of them concerns the choice of the geometry of the ambient space. Possible candidates are flat spaces, Ricci flat spaces, and constant curvature spaces. Once this geometry is chosen, we get to the second problem, which concerns the dimension and uniqueness of its metric

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signature. For example, the Schwarzschild space-time has two different flat embeddings with the same number of dimensions (six), but with different signatures. The choice of the wrong signature leads to closed timelike geodesics (Fronsdal, 1959).

Although there are some interesting aspects related to Ricci flat embedding spaces (Romero *et al.*, 1995), to constant curvature embedding spaces (Holdom, 1983), and even to spaces with dynamically variable dimensions and signatures (Regge and Teitelbolm, 1975; Ellis *et al.*, 1992), in the present note we study of the flat embedding case. This corresponds to the situation where the largest degree of freedom is obtained and where the signature ambiguity problem is fixed for true 1:1 embeddings with the least number of dimensions D. In this case we also have that the twisting vector[3] induces a connection in space-time (the twisting connection) which transforms like a gauge potential under a subgroup of the embedding symmetry (Maia and Monte, 1996).

The isometric embedding of a space-time with metric g_{ij} is given by a parametrization $\mathcal{Y}^{\mu}$ of the embedding space $\mathcal{M}_D$, and by a set of vector fields $\mathcal{N}^{\mu}_{A}$ such that[4]

$$g_{ij} = \mathcal{Y}^{\mu},_i \mathcal{Y}^{\nu},_j \eta_{\mu\nu}, \qquad \mathcal{N}^{\mu}_{A} \mathcal{Y}^{\nu},_i \eta_{\mu\nu} = 0, \qquad \mathcal{N}^{\mu}_{A} \mathcal{N}^{\nu}_{B} \eta_{\mu\nu} = g_{AB} = \epsilon_A \delta_{AB} \tag{1}$$

where $\epsilon_A = \pm 1$, $\eta_{\mu\nu}$ denote the Cartesian components of the metric of $\mathcal{M}_D$, and g^{AB} is the metric of the orthogonal subspace. Let (p, q) denote the signature of the embedding space $\mathcal{M}_D$. Since the tangent space of the embedded space-time is Minkowski's space-time [with signature (3, 1)], it follows that the subspace orthogonal to the space-time has an isometry group $SO(p - 3, q - 1)$ which is a proper subgroup of the homogeneous embedding symmetry $SO(p, q)$. The former is a semisimple group with Lie algebra generators L^{AB}. Obviously, to this Lie algebra and to g_{AB} we may associate a Clifford algebra with generators E^A and identity $E^0 = 1$ such that

$$E^{(A}E^{B)} = g^{AB}E^0, \qquad \gamma L^{AB} = [E^A, E^B], \qquad \nabla_i E^A = 0, \qquad \nabla_i L^{AB} = 0 \tag{2}$$

where γ is a proportionality factor arising from the isomorphism between the Lie algebra and the subalgebra of the Clifford algebra generated by the commutators. For convenience we introduce the Lie algebra-valued vector $A_i = A_{iAB}L^{AB}$ and the Clifford algebra-valued tensor $b_{ij} = b_{ijA}E^A$.

[3] To avoid confusion with the Einstein–Cartan geometry, we use the designation of "twisting vector" instead of "torsion vector."

[4] Lower case Latin indices run from 1 to 4 and capital Latin indices run from 5 to D. All Greek indices run from 1 to D. The indicated antisymmetrization applies only to the indices of the same kind near the brackets.

With the above notation, two of the fundamental equations of submanifolds, Gauss' and Ricci's equations, assume similar structures, namely they describe the curvature tensors of the Levi-Civita and the twisting connections respectively as algebraic functions of the extrinsic curvature b_{ij} (Maia and Monte, 1996),

$$R_{ijkl} = b_{i[k}b_{j]l} - b_{i[l}b_{j]k} \tag{3}$$

$$F_{ij} = \frac{1}{2} g^{mn} b_{n[i}b_{j]m} \tag{4}$$

Denoting the covariant derivative associated with the twisting connection by $D_i = \nabla_i - A_i$, Codazzi's equations have a different look and meaning:

$$D_{[j}b_{k]i} = 0 \tag{5}$$

In the next section we apply Frobenius' theorem to show the importance of this equation to the integrability of the two involved connections. Section 3 adapts the fundamental theorem for submanifolds to space-times and finally, in Section 4 we discuss some possible implications of this result to spin-2 fields and to geometrodynamics.

2. INVOLUTIVE CONNECTIONS

A distribution $\{\xi_i\}$ on a manifold $\mathcal{M}_D$, is said to be involutive if there are functions ϕ^k_{ij} in $\mathcal{M}_D$ such that $[\xi_i, \xi_i] = \phi^k_{ij}\, \xi_k$. The local Frobenius theorem states that an involutive distribution $\{\xi_i\}$ is also integrable (Boothby, 1975) and that in particular, an involutive distribution of independent vector fields integrates as a submanifold of $\mathcal{M}_D$ (Sternberg, 1964; Jacobowitz, 1982).

Now consider a connection Γ_i, defined in $\mathcal{M}_D$ by the respective covariant derivative $\mathcal{D}_i$. We say that the connection Γ_i is involutive if for a set of independent vector fields $\{X_k\}$ we have

$$[\mathcal{D}_i, \mathcal{D}_j]X_k = \phi^l_{ijk}X_l \tag{6}$$

where again ϕ^l_{ijk} are functions on $\mathcal{M}_D$. In general Γ_i has a curvature defined by the the Lie product $[\mathcal{D}_i, \mathcal{D}_j]$, so that the Frobenius theorem for connections is a statement on the curvature of that connection in terms of the functions ϕ^l_{ijk}. In what follows we apply this concept to the Levi-Civita and to the twisting connection of the space-time.

The two basic conditions for the existence of a four-dimensional embedded space-time in $\mathcal{M}_D$, the Gauss and Weingarten expressions, are respectively given by

$$\nabla_j \mathcal{Y}^\mu_{,i} = -g^{MN} b_{ijM} \mathcal{N}^\mu_N \tag{7}$$

$$\nabla_k \mathcal{N}^\mu_A = -g^{mn} b_{kmA} \mathcal{Y}^\mu_{,n} + g^{MN} A_{kAM} \mathcal{N}^\mu_N \tag{8}$$

Taking the covariant derivative of (7) and exchanging the indices and subtracting, we obtain

$$[\nabla_k, \nabla_j]\mathcal{Y}^{\mu}_{,i} = -2g^{MN}\nabla_{[k}b_{j]iM}\mathcal{N}^{\mu}_{N} + 2g^{MN}b_{i[jM}\nabla_{k]}\mathcal{N}^{\mu}_{N}$$

or, after using (8) and introducing the auxiliary notation

$$D^{Q}_{Mk} = \delta^{Q}_{M}\nabla_k - g^{PQ}A_{kPM} \tag{9}$$

we can write the above expression as

$$[\nabla_k, \nabla_j]\mathcal{Y}^{\mu}_{,k} = 2g^{MN}D^{Q}_{M[k}b_{j]iQ}\mathcal{N}^{\mu}_{N} - 2g^{mn}g^{MN}b_{i[jM}b_{k]mN}\mathcal{Y}^{\mu}_{,n} \tag{10}$$

As we see, ∇_i is not involutive, because the presence of the term in $\mathcal{N}^{\mu}_{N}$ in (10). Therefore, if we impose the condition

$$D^{Q}_{M[k}b_{j]iQ} = 0 \tag{11}$$

the connection ∇_t becomes involutive in the sense that

$$[\nabla_k, \nabla_j]\mathcal{Y}^{\mu}_{,i} = \phi^{n}_{ijk}\mathcal{Y}^{\mu}_{,n} \tag{12}$$

where we have denoted

$$\phi^{n}_{ijk} = 2g^{mn}g^{MN}b_{i[jM}b_{k]mN}$$

Comparing (12) with the expression for the Riemann tensor $[\nabla_k, \nabla_j]\mathcal{Y}^{\mu}_{,i} = g^{mn}R_{ijkm}\mathcal{Y}^{\mu}_{,n}$, it follows that

$$R_{ijkm}\mathcal{Y}^{\mu}_{,n} = -2g^{MN}g^{mn}b_{i[jM}b_{k]mN}\mathcal{Y}^{\mu}_{,n}$$

After replacing $g^{MN} = E^{(M}E^{N)}$ and using the notation $b_{ij} = b_{ijA}E^{A}$, we obtain Gauss' equation (3).

To understand the meaning of (11), recall that the twisting vector is a Lie algebra-defined object with an associated covariant derivative given by $D_i = \nabla_i - A_i$. For an algebraic object X, we have

$$D_iX = \nabla_iX - [A_i, X]$$

where the algebraic term $[A_i, X]$ is defined in the same algebra as that of the object X. In particular, applying this to $b_{ij} = b_{ijA}E^{A}$, we obtain, after using $[L^{MN}, E^{A}] = \frac{8}{\gamma}g^{A[N}E^{M]}$ and $D_ig^{mn} = 0$,

$$D_kb_{ij} = \nabla_kb_{ijA}E^{A} - A_{kCB}b_{ijA}\frac{8}{\gamma}g^{A[B}E^{C]}$$

Therefore, choosing $\gamma = 8$ and using (9) and (2), we can write this as

$$D_kb_{ij} = D^{A}_{Ck}b_{ijA}E^{C}$$

Consequently,

$$D_{[k}b_{i]j} = D^{Q}_{A[k}b_{i]jQ}E^{A} \tag{13}$$

Assuming (11), the right-hand side of this equation becomes zero and we obtain Codazzi's equation (5). Reciprocally, (5) implies in (11), which guarantees that ∇_i is involutive.

Next, consider the twisting covariant derivative D_i. Since $\mathcal{N}^{\mu} = \mathcal{N}^{\mu}_{A}E^{A}$ belongs to the Clifford algebra, it follows that

$$D_i\mathcal{N}^{\mu} = \nabla_i\mathcal{N}^{\mu}_{A}E^{A} - A_{iMN}\mathcal{N}^{\mu}_{A}g^{A[N}E^{M]} = D^{A}_{jM}\mathcal{N}^{\mu}_{A}E^{M} \tag{14}$$

Now, Weingarten's expression (8) with the notation (9) may also be written as

$$D^{N}_{jA}\mathcal{N}^{\mu}_{N} = -g^{mn}b_{jmA}\mathcal{Y}^{\mu}_{n} \tag{15}$$

Therefore (14) becomes

$$D_i\mathcal{N}^{\mu} = -g^{mn}b_{imM}\mathcal{Y}^{\mu}_{,n}E^{M} = -g^{mn}b_{im}\mathcal{Y}^{\mu}_{,n} \tag{16}$$

Taking the second covariant derivative of $\mathcal{N}^{\mu}$, we obtain, after exchanging $i \leftrightarrow j$ and subtracting,

$$D_i(D_j\mathcal{N}^{\mu}) - D_j(D_i\mathcal{N}^{\mu}) = [D_i, D_j]\mathcal{N}^{\mu} = 2g^{mn}\nabla_{[j}\mathcal{Y}^{\mu}_{,n}\mathrm{b}_{i]m} + 2g^{mn}\mathcal{Y}^{\mu}_{,n}D_{[j}b_{i]m}$$

and after using (7), it follows that

$$[D_i, D_j]\mathcal{N}^{\mu} = 2g^{mn}g^{MN}b_{m[iM}b_{j]n}\mathcal{N}^{\mu}_{N} - 2g^{mn}D_{[j}b_{i]m}\mathcal{Y}^{\mu}_{n}$$

Again, assuming Codazzi's equation (5) [or equivalently (11)], the last term vanishes, so that

$$[D_t, D_j]\mathcal{N}^{\mu} = 2g^{mn}g^{MN}b_{m[iM}b_{j]n}\mathcal{N}^{\mu}_{N} \tag{17}$$

In terms of components this expression is equivalent to

$$([D_i, D_j]\mathcal{N}^{\mu}_{A})E^{A} = 2g^{mn}g^{MN}b_{m[iA}b_{j]nM}\mathcal{N}^{\mu}_{N}E^{A}$$

so that, as in the case of ∇_i, the connection D_i becomes involutive only after we apply (5):

$$[D_i, D_j]\mathcal{N}^{\mu}_{A} = \phi^{N}_{ijA}\mathcal{N}^{\mu}_{N} \tag{18}$$

where we have denoted

$$\phi^{N}_{ijA} = 2g^{mn}g^{MN}b_{m[iA}b_{j]nM} \tag{19}$$

Notice that we cannot cancel $\mathcal{N}^{\mu}$ in (18) to obtain an explicit expression for $[D_i, D_j]$. This is more conveniently obtained directly from the commutator, whose components are (Maia, 1989)

$$[D_i, D_j]_{AB}L^{AB} = -(\nabla_iA_{jAB} - \nabla_jA_{iAB})L^{AB} - A_{iMN}A_{jPQ}f^{MNPQ}_{AB}L^{AB}$$

where $f_{AB}^{MNPQ} = 2\delta_A^{[N} g^{M][P} \delta_B^{Q]}$ are the structure constants of the Lie algebra of the semi-simple Lie group $SO(p - 3, q - 1)$. Consequently,

$$[D_i, D_j]_{AB} = -2(\nabla_{[i} A_{j]AB} - g^{MN} A_{[iMA} A_{j]NB})$$

Comparing the right-hand side of this expression with the left-hand side of Ricci's equation in its original form, we obtain Ricci's equation (4) in the compact form

$$F_{ij} = [D_i, D_j]_{AB} L^{AB} = 2g^{mn} b_{m[iA} b_{j]nB} L^{AB} = \frac{1}{2} g^{mn} b_{m[i} b_{j]n}$$

3. A FUNDAMENTAL THEOREM FOR SPACE-TIMES

The fact that the twisting vector A_i transforms as a gauge field suggests that it could be interpreted as Yang–Mills field of geometrical nature. To see that this is true, take the covariant divergence of F_{ij}:

$$D^i F_{ij} = g^{ik} D_k (g^{mn} b_{m[i} b_{j]n}) = \frac{1}{2} g^{ik} g^{mn} ([b_{mi}, D_k b_{jn}] - [b_{mj}, D_k b_{in}]) \quad (20)$$

Defining

$$\mathbf{j}_j^{\text{geom}} = \frac{1}{2} g^{ik} g^{mn} ([b_{mi}, D_k b_{jn}] - [b_{mj}, D_k b_{in}]) \quad (21)$$

we can write equation (20) as a Yang–Mills equation for the current $\mathbf{j}_j^{\text{geom}}$,

$$D^i F_{ij} = \mathbf{j}_j^{\text{geom}} \quad (22)$$

On the other hand, since $[D^i, [D^j, D^k]] = D^i F^{jk}$, Jacobi's identity for the Lie bracket

$$[D_i, [D_j, D_k]] + [D_k, [D_i, D_j]] + [D_j, [D_k, D_i]] = 0$$

leads to the homogeneous Yang–Mills equations

$$D^i F^{jk} + D^k F^{ij} + D^j F^{ki} = 0 \quad (23)$$

The covariant derivative of (22) gives

$$D^j D^i F_{ij} = D^j \mathbf{j}_j^{\text{geom}} = 0$$

so that $\mathbf{j}_j^{\text{geom}}$ can be interpreted as a conserved current. Therefore, the torsion vector of a space-time is a Yang–Mills field of geometrical nature with gauge group $SO(p - 1, q - 3)$, with the current given by (21). Notice that the connection A_i cannot be eliminated in the expression of $\mathbf{j}_j$, even taking into account Codazzi's equation. Consequently, the solutions of equation (22) and

(23) in general depend on an integration over some compact surface S, with an eventual association of $\mathbf{j}_i^{\text{geom}}$ with a charge of topological nature given by $q = \int_s \mathbf{j}_0^{\text{geom}}\, d^3x$.

As an example, consider Schwarzschild's space-time with the two known six-dimensional embeddings given by Fronsdal (1965) and by Kasner (1965) discussed in Maia and Monte (1996). The Kasner embedding correspond to an $SO(1, 1)$ gauge group, but this leads to noncausal situations (Fronsdal, 1965). On the other hand, taking the Fronsdal embedding, the gauge group of the twisting connection is $SO(2)$, so that A_i would be an electromagnetic field of geometrical nature. Taking the horizon of a Schwarzschild wormhole as the integrating surface, we would obtain an associated geometric charge.

To deal with Gauss' equations, it is convenient to introduce the mean curvature h and the scalar extrinsic curvature k of the space-time by

$$h^2 = g^{AB}h_A h_B, \qquad h_A = g^{ij}b_{ijA}, \qquad k^2 = g^{AB}b_{miA}b_B^{mi} = b_{miA}b^{miA} \tag{24}$$

It follows directly from the contractions of Gauss' equation (3) that

$$R_{jk} = g^{il}R_{ijkl} = 2g^{MN}g^{mn}b_{j[mM}b_{n]kN} \qquad \text{and} \qquad R = h^2 - k^2 \tag{25}$$

so that

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = \mathbf{t}_{ij}^{\text{geom}} \tag{26}$$

where we have denoted

$$\mathbf{t}_{ij}^{\text{geom}} = b_{imA}b_j^{mA} - h_A b_{ij}^A - \frac{1}{2}(k^2 - h^2)g_{ij} \tag{27}$$

Since by hypothesis our embedded manifold is a solution of Einstein's equations for a given source $\mathbf{t}_{ij}^{\text{matter}}$, equations (26) are equivalent to

$$\mathbf{t}_{ij}^{\text{geom}} = 8\pi G\mathbf{t}_{ij}^{\text{matter}} \tag{28}$$

Using the previous results, we may now announce the following theorem:

Theorem. Given a space-time of general relativity with metric g_{ij} corresponding to a source $\mathbf{t}_{ij}^{\text{matter}}$, then it has a unique flat local embedding in a pseudo-Euclidean space $\mathcal{M}_D$, with the least number of dimensions. The extrinsic curvature satisfies (5) and is related to the source by (28). Furthermore, if $D > 5$, the space-time is endowed with a geometrical Yang–Mills field A_i, with gauge group $SO(p - 3, q - 1)$, satisfying Ricci's equation (4).

In fact, assuming a true 1:1 embedding in the smallest possible space, then the signature of the embedding space is unique. If $D > 5$, and the twisting vector transforms as a Yang–Mills field with group $SO(p - 3,$

$q - 1$), according to (22) and (23), this vector satisfies the Yang–Mills equations with a current $\mathbf{j}_i^{\text{geom}}$ given by (21). On the other hand, the metric g_{ij} is the gravitational field characterized by the Einstein equations (26) with the source related to b_{ij} by (28).

Reciprocally, if we are given a tensor g_{ij}, an $SO(p - 3, q - 1)$-Lie algebra-valued vector A_i, and a Clifford algebra-valued tensor b_{ij} such that

$$G_{ij} = \mathbf{t}_{ij}^{\text{matter}} \qquad \text{and} \qquad D^i F_{ij} = \mathbf{j}_j^{\text{matter}} \tag{29}$$

then with the solutions of these equations we may write the respective curvature tensors R_{ijkl} and F_{ij}. Replacing these curvatures in equations (3), (5), and (4), we obtain the system of equations to determine b_{ij}:

$$D_{[j}b_{k]i} = 0, \qquad b_{[ik}b_{lj]} - b_{[jk}b_{li]} = R_{ijkl}, \qquad g^{mn}b_{n[i}b_{j]m} = 2F_{ij}$$

Therefore, we obtain a complete set of quantities g_{ij}, A_{iAB}, and b_{ijA} satisfying the integrability conditions for the embedding of the space-time in $\mathcal{M}_D$.

4. DISCUSSION

The results of last sections bring two new and independent entities b_{ij} and A_j into general relativity besides the metric. The extrinsic curvature b_{ij} is related to the source of gravitation by the algebraic (nondifferentiable) equations (28). If the extrinsic geometry is not fixed, we can always adjust it so as to satisfy (28). On the other hand, (28) says that for a given codimension and signature, the behavior of the extrinsic curvature may impose severe limitations to the matter content of the space-time. A simple example of this is given by a Friedman–Robertson–Walker universe with dust matter, $\mathbf{t}_{ij}^{\text{matter}} = -\rho u_i u_j$, with $g^{ij}u_i u_j = -1$, embedded in a five-dimensional flat space with signature (4, 1). In this case we have $A_i = 0$ and it follows from (28) that

$$g^{mn}b_{im}b_{jn} - hb_{ij} - \frac{1}{2}(k^2 - h^2)g_{ij} = -8\pi G\rho u_i u_j$$

so that $k^2 - h^2 = -8\pi G\rho$. This is satisfied with the extrinsic curvature given by

$$b_{ij} = \frac{1}{2}\sqrt{\frac{8\pi G\rho}{3}}\, g_{ij}$$

If ρ is constant, we obtain a constant curvature space as a consequence of (3). However, if ρ varies with time, the extrinsic curvature would be significant at times when the density was very high. In this case, the associated scalars h and k could play a role in inflation.

A general expression of b_{ij} in a five-dimensional embedding has been given by Szekeres (1966),

$$b_{ij} = \lambda g_{ij} + (4\lambda - h)u_i u_j$$

where λ is a constant. Using (28), we see that this extrinsic curvature restricts the type of matter allowed in the space-time.

With rare exceptions (Maia and Monte, 1996; Holdom, 1983), the use of the twisting vector A_i as a geometrical gauge field has not been considered in the literature. This is possibly due to the fact that the original equations for this field are not explicit. The resulting expressions (4) and (22) place the gauge field properties of the twisting vector in evidence. For example, by taking a six-dimensional embedding of the Schwarzschild space-time, we may end up with an electromagnetic field, as already mentioned. Another interesting situation occurs if we break the 10-dimensional barrier for analytic embeddings, considering the 14-dimensional case. In this case we would end up with a Yang–Mills field corresponding to a $SO(10)$ gauge group.

It is known that the Einstein and Yang–Mills equations act as integrability conditions for some linear systems (Dubois–Violette, 1983). Here we have shown that they also play a role as integrability conditions for space-times submanifolds. However, as we have seen, they are not sufficient, as the Codazzi equation (5) also plays an essential role in the application of the Frobenius theorem and it cannot be dispensed with. Only after (5) is imposed are equations (22) and (23) formally identical to the Yang–Mills equations relative to the gauge group $SO(p-1, q-3)$, whose source is derived from the extrinsic geometry of space-time. Equation (5) looks more like a constraint on b_{ij} than a dynamical equation. Since this is a symmetric rank-two tensor, we could use Gupta's theorem (Gupta, 1954; Deser, 1970) to derive a possible dynamical equation for b_{ij} as a spin-2 field. Denoting by B_{ij} the Ricci tensor of b_{ij} and $B = b^{ij}B_{ij}$, such equation would be given by an Einstein-like equation

$$B_{ij} - \frac{1}{2} B b_{ij} = 0$$

It is also possible to interpret the extrinsic curvature tensor b_{ij} as just an intermediate field between the matter sources and the gravitational and gauge fields, suggesting that this field could eventually be eliminated between the equations (22) and (26) in a higher dimensional model. Consider a situation where physics is not necessarily confined to the four-dimensional hypersurface of M_D. For example, in a high-energy process, such as particle collisions, some particles which would be otherwise constrained to the space-time hypersurface would be able to escape to move along the embedding space. This process could be associated with an alternative way to explain the nonobservability of the extra dimensions (Rubakov and haposhnikov, 1983; Visser, 1985). This may be formulated by an embedded manifold described by the coordinates

$$\mathcal{X}^{\alpha}(x^i, x^A) = \mathcal{Y}^{\alpha}(x^i) + x^A \mathcal{N}^{\mu}_A \tag{30}$$

From these coordinates we obtain the vielbein $\mathcal{X}^{\mu}_{,\alpha}$ on which the metric of M_D resembles that of Kaluza–Klein theory

$$\gamma_{\alpha\beta} = \mathcal{X}^{\mu}_{,\alpha}\mathcal{X}^{\nu}_{,\beta}\eta_{\mu\nu} = \begin{pmatrix} \tilde{g}_{ij} + g^{MN}A_{iM}A_{jN} & A_{iM} \\ A_{jN} & g_{AB} \end{pmatrix} \tag{31}$$

where we have denoted

$$\tilde{g}_{ij} = g^{mn}(g_{im} - x^A b_{imA})(g_{jn} - x^B b_{jnB}), \qquad A_{iM} = x^A A_{iMA} \tag{32}$$

The Einstein–Hilbert Lagrangian calculated with (31) gives (after using an analogy with the Kaluza–Klein metric ansatz)

$$\mathcal{R}(\gamma)\sqrt{\gamma} = \tilde{R}(\tilde{g})\sqrt{\tilde{g}} + \frac{1}{4}\,\mathrm{tr}\, F^{ij}F_{ij} \tag{33}$$

Since our space $\mathcal{M}_D$ is flat, taking variation with respect to $\tilde{g}_{ij}$ we obtain the Einstein–Yang–Mills equations for the field $\tilde{g}_{ij}$,

$$\tilde{R}_{ij} - \frac{1}{2}\tilde{R}\tilde{g}_{ij} = \mathbf{t}_{ij}(F) \tag{34}$$

where in the right-hand side we have the energy-momentum tensor of the A_i field and all contractions are made with respect to $\tilde{g}_{ij}$. Therefore the four-dimensional manifold described by (30) with metric $\tilde{g}_{ij}$ is a space-time solution of the Einstein–Yang–Mills equations (34). This looks somewhat akin to geometrodynamics (Wheeler, 1957) where the geometric electromagnetic potential is given by the twisting connection A_i and the Yang–Mills–Wheeler geons would be four-dimensional compact solutions of (34). On the other hand, we also see strong analogies with Kaluza–Klein theory, where (31) would replace the Kaluza–Klein metric ansatz and the four-dimensional space-time is described by (30).

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